The Nature and Origins of Mathematical Skills
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Chapter 12

COGNITIVE NUMBER PROCESSING: AN ENCODING-COMPLEX PERSPECTIVE

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Summary

According to the encoding-complex approach (Campbell & Clark, 1988; Clark & Campbell, 1991), numerical skills are based on a variety of modality-specific representations (e.g., visuo-spatial and verbal-auditory codes), and diverse number-processing tasks (e.g., numerical comparisons, calculation, reading numbers, etc.) generally involve common, rather than independent, cognitive mechanisms. In contrast, the abstract-modular theory (e.g., McCloskey, Caramazza, & Basili, 1985) assumes that number processing is comprised of separate comprehension, calculation, and production subsystems that communicate via a single type of abstract quantity code. We review evidence supporting the specific-integrated (encoding-complex) view of number processing over the abstract-modular view, and report new experimental evidence that one aspect of number processing, retrieval of simple multiplication facts, involves non-abstract, format-specific representations and processes. We also consider implications of the encoding-complex hypothesis for the modularity of number skills.

Introduction

In this paper we present evidence for an *encoding-complex* view of cognitive number processing (see also Campbell & Clark, 1988; Clark & Campbell, 1991). The central assumption of our approach is that number concepts and skills are based on modality- and format-specific mental codes that are interconnected in a complex and highly integrated associative structure. Furthermore, the specific symbol manipulation skills reflected in different numerical tasks (e.g., magnitude comparison, number-fact retrieval, number naming, etc.) are generally based on activation of learned associative relations involving common representational structures.

The encoding-complex view can be contrasted with *abstract-modular* theories, which view number skills as comprised of specialized processing modules that communicate by way of a single type of abstract, semantic number representation. The most developed version of the abstract-modular view of number processing is the theory proposed by M. McCloskey, A. Caramazza, S. Sokol and their colleagues (e.g., McCloskey, Caramazza & Basili, 1985; McCloskey, Sokol & Goodman, 1986; Sokol, Goodman-Schulman, & McCloskey, 1989; Sokol, McCloskey, Cohen, & Aliminosa, 1991). We will argue that the two central assumptions of the abstract-modular theory -- that the meaning of numbers is based on a single type of abstract quantity code, and that number processing entails a small number of separate "modules" -- underestimate the complexity of number processing. Because these assumptions influence the direction of theoretical inquiry, determine and validate the empirical methods used, and are at the heart of the logic by which the modularity hypothesis permits functional diagnosis of acalculia disorders, they deserve careful scrutiny.

Specific-integrated versus abstract-modular views of number processing

The model of number-processing presented by McCloskey et al. (1985) proposes three functionally distinct subsystems corresponding to number "comprehension," "calculation," and "production" modules (see also Sokol et al., 1989; Sokol et al., 1991). The primary function of the comprehension system is to recode number stimuli encountered in various formats (e.g., digits or number words) into an abstract, semantic code that is the same for all surface formats. These abstract codes provide the basis for subsequent processing. For example, the abstract codes provide input to a separate calculation system, which stores associative number facts and computes arithmetic procedures and relations. The abstract output of the comprehension and calculation systems can also be input to the verbal and Arabic production subsystems. The primary function of the production system is to translate the abstract representations of numbers into specific number codes, such as digits or written or spoken number words.

The basic assumptions of the encoding-complex view are incompatible with this strong form of modularity. Instead, we propose that the collections of tasks denoted by the terms comprehension, calculation, and production, primarily utilize common cognitive representations and resources (e.g., visual and verbal working-memory systems), although the tasks may differ to some extent in terms of specific processing and response requirements. In other words, number comprehension, calculation, and production are coextensive skills rather than separate processing modules.

The emphasis on abstract codes in the McCloskey et al. model also is incompatible with the encoding-complex approach. Specifically, McCloskey et al. (1986; Sokol et al., 1989) claim that different surface forms of numbers (e.g., digits and number words) are translated into a uniform modality-free semantic code that entails an abstract representation for each number, plus the power of ten appropriate to its place value. In their notation, the quantity *ninety*, for example, is represented semantically by the abstract code {9}10EXP1, and the quantity *eight* by {8}10EXP0, irrespective of the manner in which the quantity was encoded (e.g., spoken or visually presented) or computed. In contrast, the most basic assumption of the encoding-complex view is that numbers are represented in terms of modality-specific mental codes. Verbal or word codes include articulatory and auditory codes in most people, visual and written number-word codes in literate individuals, and unique codes in various specific groups (e.g., sign-language codes for numbers). In addition to verbal codes, number processing implicates such nonverbal codes as visual and written codes for digits, imaginal, analogue codes for magnitude (e.g., number lines), and combined visual-motor representations (e.g., counting on fingers; using an abacus). Being associatively connected, the various specific codes can activate one another to produce a multi-component representational structure that we call an encoding complex.

In this paper, we review evidence supporting a role for modality-specific number representations (for a more thorough discussion see Clark & Campbell, 1991), and report a study of simple multiplication showing clearly that retrieval varies with the surface form in which a problem is presented. We then shift the focus to the hypothesis of modularity in number processing, and argue that the sharp separation of number comprehension, calculation, and production in the abstract-modular theory fails to recognize the strong interdependence and relatedness of these facets of number processing.

Specific versus abstract number codes

A variety of internal codes appears to be necessary to explain how people represent number words, digits, and other quantitative stimuli. For example, many number-processing tasks require subjects to temporarily store information while performing some other mental operation (e.g., Campbell & Charness, 1990). This storage of intermediate results implicates working-memory processes, which are assumed to involve temporary retention of verbal or visuo-spatial representations (Baddeley, 1986; Baddeley & Hitch, 1974). The disruptive effects of concurrent articulation in mental addition (Hitch, 1978) and counting tasks (Logie & Baddeley, 1987; see also Healy & Nairne, 1985) implicates verbal working-memory processes (i.e., phonological and articulatory codes) in mental arithmetic. Other research demonstrates the importance of visual-spatial codes in working memory (Baddeley, 1986), and also provides evidence of visuo-spatial codes in calculation. For example, Stigler (1984; see also Hatano, Miyake, & Binks, 1977) proposed that skilled users of the "mental abacus" employ a visual representation of the abacus to perform mental arithmetic, and Hayes (1973) obtained evidence for the use of visual imagery in a large number of simple calculation tasks. Beyond evidence for visual and verbal number codes in working memory, experimental research also implicates modality- or format-specific processes in a variety of basic number comprehension and calculation tasks.

Format-specific phenomena in number processing

Effects of surface format (e.g., digits vs. number words) on number processing potentially contradict the abstract-code hypothesis, because processing based on abstract codes should be the same irrespective of input format. In contrast, the encoding-complex view implies that different surface forms can vary in their capacity to activate the specific internal codes that mediate performance, and therefore the representational basis of performance can change with changes in surface form.

Clark and Campbell (1991) describe a substantial body of evidence indicating that the internal mechanisms mediating performance in a variety of number tasks varies with the format of stimulus presentation. For example, Vaid and Corina (1989) demonstrated that both stimulus format and prior experience determine how numerical comparisons are performed (see also Foltz, Poltrock & Potts; 1984; Tzeng and Wang, 1983). Vaid and Corina compared Stroop interference in speeded magnitude judgments between digits, English words, and number signs in American Sign Language (ASL) - English bilinguals. Stimuli were presented either in the left-visual-field (LVF) or the right-visual-field (RVF). Overall, there was greater interference when number words and signs were initially processed in the RVF, whereas interference was greater for digits presented to the LVF (see also Vaid, 1985). Moreover, for number words and signs, interference was greater in the RVF for the more skilled language (i.e., English or ASL), but was greater in the LVF for the less skilled language.

Vaid and Corina concluded that magnitude judgments about digits versus words involve internal representations that are differentially accessed in the right and left hemispheres respectively, and that degree of language skill affects lateralization of magnitude judgments. Specifically, magnitude judgments for number words appear to be more dependent on left-hemisphere functions in right-handers. Thus, processing of numerical magnitude appears to depend on format, apparently because left-hemisphere codes (e.g., possibly verbal representations) and right-hemisphere codes (e.g., visuo-spatial representations) may be differentially engaged in judgments about magnitude as a function of format (e.g., digits vs. number words). Clark and Campbell (1991) also reviewed evidence that processing of odd-even status varies with stimulus format (e.g., Hines, 1990; Klein & McInnes, 1988; Shepard & Gale, 1982), which supports the view that knowledge and use of the odd-even relationship is also mediated by multiple forms of mental codes.

Effects of number format are particularly relevant in the present context because proponents of the abstract-modular theory use format-based dissociations in brain-damaged patients as a primary method for deducing the architecture of number processing and for inferring the functional locus of a patient's deficit. The theory proposed by McCloskey et al. (1985) makes a strong distinction, for example, between comprehension deficits (i.e., failure to access the correct semantic-quantity code) and calculation deficits (i.e., failure to perform the correct computation given input of the appropriate abstract code). For example, if a calculation can be performed successfully given arabic digits, but cannot be performed when the stimuli are presented using number words, then one can conclude that the comprehension subsystem for words must be impaired. This follows because unimpaired calculation for digits confirms that the abstract calculation process is intact (cf. McCloskey et al., 1985, p. 178). This logic is valid, however, only when it is assumed that processes "down stream" of stimulus encoding are constant irrespective of format. If different number formats can differentially activate internal number representations and associations that are functional in performance, as the encoding-complex hypothesis proposes, then mechanisms of numerical judgment, calculation, and production can vary as a function of number format. This would, at the very least, restrict the generality of using format-related dissociations to infer the functional locus of a numerical deficit.

It is important to note that the critical issue is not whether abstract number codes exist, but rather whether calculation and production can be mediated by multiple codes whose roles vary with format (e.g., visual codes may be especially salient with digit stimuli, whereas activation of phonological codes may be more salient with number words). If such is the case, then the logic for interpreting performance dissociations described above is invalid whether or not some form of abstract code also exists.

An experimental demonstration of format effects in basic arithmetic

In the following sections, we present experimental evidence that retrieval of simple multiplication facts (e.g., 3 x 6 = ?) is sensitive to number format (digit vs. number-word stimuli). These data have important implications for the abstract-code hypothesis because number-fact retrieval unambiguously implicates the abstract calculation module in the McCloskey et al. (1985; Sokol et al., 1989; Sokol et al., 1991) model and thus should not be sensitive to surface format. In contrast, the encoding-complex assumption of modality-specific codes implies that retrieval processes can be format specific. For example, digits and number-words could differ in their capacity to activate visuo-spatial and verbal-code representations of problems. This multiple-code view is consistent with the findings of Kashiwagi, Kashiwagi et al., (1987), who studied Japanese aphasics with impaired performance for simple multiplication. Despite extensive practice, the patients could not relearn multiplication with verbal presentation and responses. They did, however, learn to generate the multiplication facts given visual presentation combined with written responses. Such findings support the theory that the representations underlying retrieval of number facts can involve multiple codes that are differentially involved as a function of surface form.

Although there are performance effects due to format, some researchers have concluded that these effects could be due to differences in encoding stimuli in different formats (e.g., Marsh & Maki, 1976; McClain & Shih Huang, 1982). Based on such possibilities, proponents of the abstract-modular view have argued that results such as those of Kashiwagi et al. (see also, Gonzalez and Kolers, 1982; 1987) are not compelling with respect to the issue of format-specific processes in calculation (Sokol et al., 1989; Sokol et al., 1991).

To clarify this issue, the following study examined number-fact retrieval (simple multiplication) with digit and word stimuli. In particular, we examined whether surface format interacted with problem-size and other indices of retrieval difficulty. Such interactions would support the specific-code hypothesis that number-fact retrieval can involve format-specific representations. We also examined error patterns as a function of presentation format. A previous study by Sokol et al. (1991) examined format effects in the simple-multiplication errors of a single acalculia patient, PS, and found similar error patterns when digits, number words, or dots were used to represent numbers. The current study, which involved a large number of normal adult subjects, may permit a more sensitive analysis. Finally, we conducted reanalyses of the simple multiplication errors of Sokol et al.'s (1991) patients GE and PS, and further analyses of the present multiplication experiment, that suggest that number-reading and number-fact retrieval are integrated, interactive processes, rather than functionally independent processes as the abstract-modular theory assumes.

Method

The subjects were 80 undergraduate students (36 females, 44 males) at the University of Western Ontario who ranged in age from 18 to 28 years. Stimuli were the multiplication problems in the range from 2 x 2 through 9 x 9 presented in digit format or in English number-word format using either upper or lower case letters. Problems were presented horizontally with the two operands separated by a lower case "x" with flanking spaces. Digit-format problems were 2 centimeters in length whereas word-format problems ranged in length from 2.5 to 3.5 centimeters.

Subjects received four blocks of 62 trials, with word format used for odd-numbered trials and digit-format for even-numbered trials. The problems tested yield 31 different products, with five of the products being correct answers to two combinations of operands (i.e., 12, 16, 18, 24 and 36). The 31 digit and 31 word trials in each block included the same shared-product problems, and both sets included problems involving all 31 products. The five shared-product problems excluded in the first block were exchanged for their same-product counterparts in Block 2. Blocks 3 and 4 replicated the problems in Blocks 1 and 2, respectively. The specific set of shared-product problems tested in the first block was counterbalanced within each of two sets of 32 subjects, and was chosen arbitrarily for the remaining 16 subjects. For each subject the order of problems in each block was pseudo-random, with the constraint that the digit trial and the word trial involving the same correct product were separated by at least 20 trials within a block. The order of operands for non-tie problems was determined randomly and independently for word and digit problems in Block 1 and then alternated across blocks. Subjects were instructed to state the correct answer to each problem as quickly and accurately as possible.

A computer presented the stimuli and recorded response times (RTs) to ± 1 ms. For each trial, the prompt "words" or "digits" appeared briefly at the center of the computer screen, and was followed by a fixation dot for 1.5 s. The problem then appeared with the multiplication sign at fixation. Timing began when the problem appeared and was stopped when the subject's spoken response triggered a voice-activated relay. An experimenter recorded the response given on each trial.

Effects of format on speed and accuracy of simple multiplication

RT summary. Table 1 presents mean correct RTs for "easy" and "difficult" problems presented in digit and word formats[1]. Problem difficulty was determined from the table of normative multiplication performance reported by Campbell and Graham (1985, Appendix B) and the assignment of problems to easy and difficult cells was the same as that used by Campbell (1987; 1991). A two-by-two repeated-measures analyses of variance (ANOVA) with factors of format and difficulty confirmed that the difficult problems produced longer RTs relative to the easy problems, $F(1, 79) = 210.3$, $MSe = 24727.6$, $p < .001$, and showed that word problems required longer RTs than digit problems, $F(1, 79) = 513.3$, $MSe = 13261.3$, $p < .001$. Although the pattern of RTs across the set of 64 problems was very similar for digit and word formats ($r = .913$), the interaction of format and difficulty was highly significant, $F(1, 79) = 101.4$, $MSe = 2710$, $p < .001$. The interaction shows that the increase in RT with words relative to digits was greater for the difficult problems.

Table 1. Mean RT and %E for Easy and Difficult Problems as a Function of Presentation Format

Problem Type	Words	Digits
	RT (SD)	
Easy	950 (147)	717 (96)
Difficult	1264 (289)	914 (220)
	1107	816
	%E (SD)	
Easy	5.0 (6.2)	2.1 (2.1)
Difficult	19.9 (12.5)	14.5 (10.1)
	12.5	8.3

Note. $n = 80$ subjects.

[1] There were no apparent effects of upper vs. lower case letters and this factor is not considered in the following analyses.

Error summary. Subjects made a total of 861 errors with digits and 1329 errors with words, an increase of 54% from digit to word format. The difference between mean errors per subject for digits (10.8) and words (16.6) was highly significant, $SE = 1.02$, $z = 5.71$, $p < .001$. The majority of errors were commission errors for both words (1236 errors or 93%) and digits (822 errors or 95%). Although commission errors accounted for the bulk of errors, omission errors (i.e., failures to produce an intelligible response before triggering the voice key) were much more common with the word format (93 errors) than the digit format (39 errors), an increase of 138% versus the 50% increase in commission errors. The mean number of omission errors per subject was 1.16 for words and 0.49 for digits, $SE = .23$, $z = 2.96$, $p < .001$.

A two by two (Easy vs. Difficult by Words vs. Digits) repeated-measures ANOVA on the mean rates of commission errors (see Table 1) confirmed a higher error rate for word problems than digit problems, $F(1, 79) = 28.4$, $MSe = 49.1$, $p < .001$, and verified a higher error rate for difficult problems relative to the easy problems, $F(1, 79) = 220.8$, $MSe = 68.1, p < .001$. The interaction of format and difficulty, [$F(1, 79) = 8.3$, $MSe = 15.1, p = .005$,] indicated that the increase in errors due to the word format was greater for the more difficult problems, although error rates across problems in each format were highly correlated ($r = .895$).

The encoding-complex view accounts for format-related differences in RTs and errors in terms of the distinct numerical processing associated with digit and word formats, including differential access to number facts, format-specific generalization effects to related number codes, and distinct sorts of response priming effects. For example, multiplication facts may involve associative networks among digit-like codes, which are more readily accessed by the digit format than by the word format. Weaker word-format associations or increased word-format competition could readily produce increases in RTs and both commission and omission errors. Several of these mechanisms are described more fully in the following sections of the paper. To begin, we examine the patterns of RT and errors across problems and format in more detail using multiple regression techniques. Based on the these results, we argue that format is directly influencing associative retrieval processes. Subsequent to this, we show that format produces robust effects on the patterns of specific multiplication errors, and we argue that these effects further support the encoding-complex approach and challenge the abstract-modular theory.

Multiple regression analyses of RTs and errors

The dependent variables for the multiple regression analyses were the mean correct RT and commission-error rate for each of the 64 problems in each format. Only a subject's first correct trial in each format for each problem contributed to the RT means. There were four predictor variables used in the regression: 1) problem *Size* (answers < 20 coded as -1, ≥20 and <40 coded 0, ≥40 coded +1), 2) *Fan* (shared-product problems [n = 18] coded +1 and unique-product problems [n = 46] coded 0), 3) *Length* (the number of character spaces occupied by a problem in the word format), and 4) *Ties* ("tie" problems such as 2 x 2, 3 x 3 coded +1 and non-ties as 0). Length and Ties were included to assess possible contributions of stimulus reading or encoding factors to between-format differences. Size provides an index of problem difficulty (e.g., Campbell & Oliphant, this volume) that is not confounded with the tie/non-tie factor (r = .03). In contrast, the breakdown into "easy" and "difficult" multiplication problems used by Campbell (1987, 1991) places all but one of the tie problems (8 x 8) into the easy set. Shared-product problems might show a *fan effect*, an interference phenomenon apparently due to the number of irrelevant associations "fanning out" from a concept in memory (e.g., Pirolli & Anderson, 1985); thus, Fan was assumed to potentially index a source of retrieval interference. In this case, problems with shared products should tend to be more difficult, once other factors contributing to difficulty (i.e., factors estimated by product size) are taken into account.

Table 2 presents the results of separate multiple regression analyses of the word-format data, digit-format data, and the differences between formats (word data minus digit data). A forward-stepwise procedure was used for entering the predictors (Size, Fan, Ties, and Length) into the equations. The criterion for a variable's entry was that beta differed from 0 with $\alpha < .05$.

Word format. In the regression of word-format RTs, all four variables entered the equation and produced an R^2 of .710 [$F(4, 59) = 39.57$, $MSe = 10070.90$, $p < .001$].[2] In the analysis of word-format errors, Size, Fan, and Ties each accounted for significant variability [$R^2 = .493$, $F(3, 60) = 19.47$, $MSe = 56.61$, $p < .001$]. Problem length apparently did not affect errors in the word-format condition, suggesting that the relation between Length and word-format RT primarily reflects stimulus reading time rather than a general encoding difficulty for longer number words. The overall performance advantage for tie problems is a well-established effect in both multiplication and addition (e.g., see Campbell & Oliphant, this

[2] All R^2s are adjusted for the number of predictors in the equation.

volume). Fan entered in both the RT and error analyses with positive beta weights indicating that the shared-product problems tended to be slightly more difficult once other difficulty factors were partialled out.

Table 2. Raw Regression Weights from Problem-based Multiple Regression Analyses of RT and %E

	Words	Digits	Words–Digits
Predictors		RT	
Size	159.6***	124.8***	36.7***
Fan	86.1**	46.2*	38.1*
Ties	-268.0***	-93.1***	-175.1***
Length	32.7**	---	27.6***
Intercept	942.5	873.7	111.7
		%E	
Size	8.9***	6.7***	1.5*
Fan	6.4**	---	3.9**
Ties	-9.7***	-7.6**	---
Length	---	---	---
Intercept	13.4	10.1	3.7

Note. *** = $p < .001$, ** = $p < .01$, * = $p < .05$. Variables that did not reach the .05 level of significance were not included in equations. R^2's adjusted for number of predictors. $n = 64$ problems.

Digit format. The regression indicated significant effects of Size, Fan, and Ties on digit-format RTs [$R^2 = .660$, $F(3, 60) = 41.80$, $MSe = 4909.00$, $p < .001$]. Length (i.e., the number of characters in the corresponding word-format problem) was not a significant predictor of digit-format RTs. In the analysis of digit-format errors, Size and Ties entered the equation [$R^2 = .454$, $F(2, 61) = 27.18$, $MSe = 40.91$, $p < .001$]. Neither Length nor Fan entered in the error analysis, although the partial correlation for Fan was in the expected direction (.163), despite a negative zero-order correlation (-.153).

Word format minus digit format. Across the 64 problems the RT and error differences as a function of format were significantly correlated ($r = .383, p < .01$), suggesting that format effects on RT and errors were mediated to some extent by common factors. In the analysis of RT differences, the coefficients for Size, Fan, Ties, and Length were all significant [$R^2 = .624, F(4, 59) = 27.19, MSe = 3116.69, p < .001$]. Specifically, the magnitude of the word-format deficit increased with problem size and tended to be larger for the shared-product problems. These factors emerged over the tendency for RT differences to increase with the length of word problems (Length) and to be reduced for problems with repeated multipliers (Ties). In the analysis of error differences, Size and Fan each accounted for independent variability, [$R^2 = .103, F(2, 61) = 4.63, MSe = 19.33, p = .013$].[3] Because Length and Ties were not significant predictors of error differences, the prediction of RT differences by Length and Ties likely reflects only differences in time to scan or read number words relative to digits, as opposed to processes related specifically to retrieval of multiplication facts.

The preceding analyses show that format had substantial effects on performance, and the interactions of format with problem size or difficulty suggest, more specifically, that retrieval processes varied as a function of format. Before discussing the implications of these findings in more detail, we present results of detailed analyses of the specific errors produced in each format. Format effects on the patterns of specific errors also support our view that calculation processes in arithmetic depend on the format of the problem.

Effects of format on specific errors

Table 3 shows commission errors divided into several mutually exclusive categories, which are described in order of classification priority. An error was classified as a *cross-operation error* if the response was the correct answer for the corresponding addition problem. A *naming error* occurred if the response consisted entirely of one or both of the problem's operands (4 x 8 = 8 or 4 x 8 = 48). Errors not classified as cross-operation or naming errors were classified as *table-related* if the error was a correct answer to another single-digit multiplication

[3] In a more recent study comparing word and digit multiplication, effects associated with the Fan variable did not emerge clearly as in the present study. This discrepancy may be due to procedural differences between the experiments; specifically, in the present study, problems sharing the same product were exchanged on alternating blocks of trials so that each product would be encountered only once within a block. In the more recent study, different problems with the same product were allowed to recur within the same block.

problem in the same times table (4 x 8 = 36). A *table-unrelated* error was a correct answer to a single-digit multiplication problem in a different times table (4 x 8 = 42). *Miscellaneous* errors were the remaining commission errors that did not fall into any of the preceding categories (4 x 8 = 34).

Table 3. Rates of Simple-Multiplication Error Types for Number-word and Arabic-Digit Presentation Formats

	Format			
	Words		Digits	
Error Type	*f*	*p*	*f*	*p*
Cross operation	21	.02	22	.04
Naming	90	.05	16	.02
Table Related	875	.72	621	.77
Table Unrelated	145	.12	110	.12
Miscellaneous	105	.09	53	.05
Total	1236	1.00	822	1.00

Note. f = frequency. p = mean proportion of commission errors per subject. $n = 80$.

Cross-operation errors were relatively infrequent and occurred with equal frequency in each format (22 vs. 21), although there was weak evidence that they accounted for a higher proportion of digit-format errors (mean proportion per subject of .02 for words versus .04 for digits, $SE = .012$, $z = -1.92$, $p = .055$). This latter effect presumably reflects the increased incidence of other errors with the word format.

The most dramatic increase in error types occurred for naming errors, which were 463% higher for words (90 errors) than digits (16 errors). Mean number of naming errors was higher for words (1.13) than digits (.20), $SE = .246$, $z = 3.76$, $p < .001$, as was mean proportion of naming errors (.06 versus .03), $SE = .012$, $z = 2.40$, $p < .02$. Naming errors are related to another phenomenon, operand intrusions, and are discussed in the next section. Both naming and operand-intrusion errors implicate format-sensitive interactions of number reading and number-fact retrieval processes that challenge the abstract-modular view.

Miscellaneous errors occurred 98% more often with word than digit format (105 vs. 53 errors). The mean frequency of these errors was higher for words (1.31) than digits (.66), $SE = .178, z = 3.65, p < .001$, as was the mean proportion (.09 vs. .05), $SE = .012, z = 3.10, p < .001$. Although we have no specific explanation for miscellaneous errors, their increased incidence supports the conclusion that word format problems introduce elements into mental multiplication that are not present with the digit format.

The percentage increase in errors was higher for naming and miscellaneous errors than for errors from the times-tables (table-related plus table-unrelated). Nonetheless, the table errors were the most common errors and increased a robust 40% from digits (731 errors) to words (1020 errors). Table-related products accounted for the largest part (61%) of the increase in errors for word relative to digit problems, and the mean number of table-related errors per subject was higher in the word format (10.9 with words vs. 7.8 with digits; $SE = .663, z = 4.79, p < .001$).

Further analyses of table-related errors pointed to additional specificity in the higher error rate for the word format, and suggested one factor that might contribute to the effect of format on table-related errors. In Table 4, table-related errors are classified according to whether the error was related to the minimum operand only (e.g., 3 x 7 = 24), to the maximum operand only (3 x 7 = 28), or to both operands (4 x 8 = 24). As Table 4 shows, the higher frequency of tabled-related errors was due mainly to an increase in errors related to the maximum operand (352 for words vs. 197 for digits), an increase of 79%. Errors related to the minimum operand demonstrated only a 16% increase. Indeed, the mean number of max-related errors was substantially higher with the word format (4.4 for words vs. 2.5 for digits, $SE = .41, z = 4.74, p < .001$), whereas the evidence was less clear that the mean number of min-related errors differed between formats (5.0 for words vs. 4.3 for digits, $SE = .36, z = 1.88, p = .06$).[4] The mean proportion of tabled-related errors that were related to the maximum operand also was higher with the word-format condition than with digits (.43 vs. .32, $SE = .034, z = 3.07, p < .001$). Errors related to both operands, an error

[4] In their following commentary on this chapter, McCloskey, Macaruso, and Whetstone correctly point out that the higher proportion of max-related errors with words is an artifact of the higher rate of *intrusion errors* observed with the word format (see below). This occurs because an operand intrusion in the units position (e.g., 9 x *6* = 3*6*) can yield an error that is either min-related or max-related, whereas an intrusion into the decade position (e.g., *6* x 9 = *6*3) can only be max related. Thus, the higher rate of intrusions for word-format problems, relative to digit problems, also results in relatively more max-related errors.

Table 4. Rates of Errors Related to the Minimum Operand Only, the Maximum Operand Only, or Both Operands

	Format			
	Words		Digits	
Related Operand	*f*	*p*	*f*	*p*
Minimum	398	.44	344	.54
Maximum	352	.43	197	.32
Both	125	.13	80	.14
Total	875	1.00	621	1.00

Note. f = frequency. p = mean proportion of table-related errors per subject. n = 80.

Table 5. Rates of Table-Related Errors as a Function of Format and Distance of the Unrelated operand

	Format			
	Words		Digits	
Distance	*f*	*p*	*f*	*p*
±1	480	.56	408	.68
±2	293	.33	173	.24
>2	102	.11	40	.08
Total	875	1.00	621	1.00

Note. f = frequency. p = mean proportion of table-related errors per subject. n = 80.

type that cannot be unambiguously associated with the minimum or maximum operand, increased by 56%.

A second phenomenon that demonstrates the specificity of the increase in table-related errors concerns the distance between the correct and incorrect operands. Table 5 shows table-related errors as a function of the "distance" of the unrelated operand (±1, ±2, > 2). For example, the error 3 x 7 = 28 corresponds to a distance of 1, because replacing 3 with 4 (a difference of 1) yields the error response of 28. The error 4 x 9 = 28 represents a distance of 2, because replacing the 9 with 7 produces 28. The data in Table 5 show a systematic increase with distance in the magnitude of the format effect. Table-related errors increased only 18% for adjacent operands (480 vs. 408 errors), 69% for a distance of two (293 vs. 173 errors), and 155% for greater than two (102 vs. 40 errors). The mean proportion of related errors greater than or equal to two operand units in distance was substantially higher with the word problems (.44 for words vs. .32 for digits, $SE = .031, z = 3.81, p < .001$).

Interpretation of the preceding table-related effects is presently very tentative. One explanation is that number-fact retrieval with word stimuli is less sensitive to numerical magnitude or proximity than with digit stimuli. The reasoning is that max-related errors, on average, will be numerically more distant from the correct answer than will be min-related errors. A similar explanation could apply to the operand distance effect in Table 5, inasmuch as errors are determined by more remote digits in the same times-table. A related possibility for the operand distance effect is that number words develop more remote associations than digits because of counting by twos and related verbal operations.

Although our explanation for the preceding effects, many of which are new, is tentative, the preceding results show definitely that the pattern of multiplication errors varies substantially between the word and digit formats, including differences in the absolute and relative frequencies of overall errors, specific categories of errors, and numerical distance effects within categories. Such findings challenge the assumption that number-fact retrieval is mediated only by a single type of abstract code that is the same for digit and word-based stimuli, and instead suggest that the internal representations mediating retrieval vary with surface form. In the following section, we report further evidence that format affects performance in additional ways, and we argue that these findings support the encoding-complex hypothesis and challenge the abstract-modular view.

Operand intrusion errors and effects of operand order

As noted earlier, subjects were much more likely to produce naming errors (e.g., 2 x 9 = 9) for words than digits. Only answers that resulted entirely from operand naming were considered naming errors, but only one of the operands was incorporated into many other incorrect responses. Campbell and Oliphant (this volume), for example, observed that about 37% of adult's errors on simple multiplication problems incorporated one of the problem's operands (e.g., 9 x *6* = *56*, *4* x 8 = *24*, *7* x 9 = *72*). Consistent with the naming error data, the mean number of operand intrusions was substantially higher for word-format problems (7.5) than for digit-format problems (4.1), SE = .580, z = 5.84, p < .001). The mean proportion of errors involving an operand-answer match also was higher with words (.51) than with digits (.38), SE = .034, z = 3.77, p < .001).

The high incidence and significant format effects for operand intrusion errors suggest that such errors reflect systematic numerical processes, rather than random factors. This inference is further confirmed by examination of operand order effects on intrusions. By chance, operand intrusions should match the left-right position of the operand only 50% of the time (see also Campbell & Graham, 1985). Campbell and Oliphant found, however, that 68% of intrusions matched the position of the "intruding" operand in the problem (e.g., 9 x 6 = 56 occurred more frequently than 6 x 9 = 56,). In the present data, the mean proportion of intrusions with position matched was .58, which is significantly greater than the .50 expected by chance (SE = .023, z = 3.36, p < .001)[5]. The effects of operand order on intrusions confirms that matches between operands and components of error responses cannot be attributed to chance.

Another interesting effect was that operand intrusions frequently co-occurred with associatively related error responses. Specifically, table-related errors constituted 77% and 79% of intrusion errors observed with the word format and digit formats, respectively. Together these findings suggest that operand-intrusion errors result from a confluence of number-fact retrieval and number reading processes.

The encoding-complex hypothesis explains operand intrusions and the specific effects cited here in a straightforward manner. Operand intrusions occur because operand-reading processes compete with fact-retrieval processes. This explanation follows from the encoding-complex assumption that number-reading and number-fact retrieval are integrated functions, utilizing common, specific representations. For example, operand reading presumably activates a variety of visual and verbal codes that are also the primary media of number-fact representation. Operand reading therefore primes retrieval of number-facts that match the problem's operands. The high proportion of tabled intrusions occurs simply because it is representations of multiplication facts that are being primed, as opposed to post-retrieval, verbal-output codes. The effects of operand order on intrusions may be attributed to feature-matching processes that activate memory representations for arithmetic facts. Given the encoding-complex assumption that

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[5] Errors on tie problems were excluded for this analysis.

memory codes for number-facts preserve perceptual or "physical" features, it is reasonable to propose that a problem representation will be activated more strongly when its features are matched in the same relative positions (cf. Campbell & Oliphant, this volume). Thus, under this encoding-complex view, operand intrusions are properly considered retrieval phenomena.

The standard abstract-modular view appears strained to explain operand-intrusion phenomena and, simultaneously, to maintain the integrity of the discrete modules and the fundamental role of the abstract codes. Although the basic fact of operand intrusions might be explained by abstract outputs of the number-comprehension system (i.e., operand codes) and abstract outputs of the calculation system (i.e., answer codes) being combined or summed in the verbal-number production module, this post-retrieval model fails to explain why operand intrusions generally involve table-related answers as opposed to random concatenations of operands with answers. That is, simple post-retrieval summation of answers and operands ought frequently to produce non-tabled answers. The abstract-modular view also has serious difficulty explaining format and operand order effects.

Format effects on intrusion errors are awkward for the abstract-modular view because digit and word naming both involve comprehension and production processes mediated by the identical abstract code. That is, naming a digit involves translating the digit into an abstract code and then translating the abstract code into a number name. Naming a number-word is also mediated by the same abstract codes. Since production in both cases is mediated by the same abstract code, the greater incidence of operand intrusions given words has no explanation within the standard components of the abstract-modular theory of number processing. Although the abstract-modular view could incorporate additional direct connections between number word stimuli and their names, such connections would bypass the abstract quantity codes and considerably weaken the ideal modular structure that is central to the abstract-modular view. A second possibility is that abstract codes are activated for the individual operands and for the combined operands (e.g., 4 x 8 is encoded both as separate operands and as 48). It seems unusual, however, that number words would be more likely than digits to activate these composite abstract codes, if in fact abstract codes are neither word-like nor digit-like.

The tendency for operand intrusions to appear in the same relative position further complicates the abstract-modular view. From that perspective, operand order effects suggest that the left operand is processed by the comprehension and production system as if it represented a tens value, whereas the right operand is processed as a units value (e.g., 4 x 8 would activate "forty eight" in the production system). But tens and units values are in fact represented by the abstract quantity codes, which we just noted may need to be bypassed to explain the greater incidence of operand intrusions for word problems. If abstract quantity codes are not involved in intrusion errors, then some independent mechanism is necessary to explain operand order effects. Although distinct from the abstract codes, the proposed mechanism must nonetheless be intimately related to calculation processes in order to explain why the majority of intrusions (about 80%) are answers in the correct times tables.

To further test the encoding-complex view that operand-intrusions reflect convergent associative retrieval processes, we examined multiplication problems in which one of the operands occurs in the correct answer. By analogy to operand-intrusion errors, when the position of an operand matches the same number in the correct answer (e.g., 6 x 4 = 24), performance should be faster and more accurate relative to when there is a positional mismatch (e.g., 4 x 6 = 24). This prediction follows from our assumption that operand reading directly primes answer representations that are the objects of number-fact retrieval. Furthermore, since word-stimuli produced a higher rate of operand intrusions, it follows under these assumptions that the facilitation due to a match between operand and correct-answer should be stronger with words than digits. No such predictions follow from the abstract-modular hypothesis that operand intrusions reflect convergence of activation at a post-retrieval, verbal-production stage.

Table 6 presents mean RT and error rates based on the twelve multiplication problems tested in which an operand appears in the correct answer (i.e., 6 x 2 = 12, 2 x 6 = 12; 3 x 5 = 15, 5 x 3 = 15, 6 x 4 = 24, 4 x 6 = 24, 7 x 5 = 35, 5 x 7 = 35, 9 x 5 = 45, 5 x 9 = 45, 6 x 8 = 48, 8 x 6 = 48). The values in the Table contrast performance on these problems when an operand matches the corresponding number in the same position in the correct answer (e.g., 6 x 4 = 24), versus when there was a positional mismatch (4 x 6 = 24).

Repeated-measures ANOVAs indicated that for this subset of items, word-format produced longer RTs [$MSe = 711$, $F(1, 5) = 960.57$, $p < .001$], and more errors [$MSe = 14.03$, $F(1, 5) = 8.19$, $p = .035$] than digit-format. More importantly in the present context, when there was a positional match, correct RTs were faster [$MSe = 1699.7$, $F(1, 5) = 9.27$, $p = .029$], and there were fewer errors [$MSe = 7.10$, $F(1, 5) = 13.96$, $p < .013$]. The accuracy advantage with an operand-answer match relative to a mismatch appeared to be somewhat stronger for word stimuli [$t(5) = 2.76$, $p < .05$] than for digit stimuli [$t(5) = 1.87$, $p < .10$], although the interaction was not significant [$MSe = 10.72$, $F(1, 5) = 1.03$]. In the RT analysis, there was a 69 ms advantage for a positional match with the word stimuli [$t(5) = 4.50$, $p < .005$], compared to a 33 ms advantage in the digit condition [$t(5) = 1.58, p < .10$], and the interaction effect approached conventional significance levels [$MSe = 338.54$, $F(1, 5) = 5.69$, $p = .063$]. Thus, there was strong evidence that operand position influenced the probability and speed of correct responding, and also weaker evidence that operand-answer priming tended to be stronger with the word format.

Table 6. Performance Differences as a Function of a Positional Match Between a Problem Operand and a Numeral in the Correct Answer (e.g., 6 x 4 = 24 vs. 4 x 6 = 24)

	RT(ms)			p(Error)		
	Match	Mismatch	Diff.	Match	Mismatch	Diff.
Digits	859	892	33	0.07	0.10	0.03
Words	1178	1247	69	0.10	0.15	0.05
	1018	1070	51	0.085	0.125	0.040

Note. RT = mean correct response time. p(Error) = mean proportion of trials that were errors. Means based on six problems.

Evidence for operand-answer priming phenomena in acalculia

The relevance of the preceding findings to the abstract-modular view would be further established by signs of operand-related phenomena in the performance of acalculia patients, inasmuch as acalculics have provided a substantial part the evidence for the abstract-modular theory. The multiplication errors produced by two acalculia adults (PS and GE) described previously by Sokol et al. (1991) indeed contain clear evidence of operand-answer priming effects. We refer the reader to the original article for details of these patients' histories and testing.[6]

PS produced a total of 214 (11.5%) incorrect responses over 23 blocks of testing on the multiplication problems from 1 x 1 to 9 x 9. Of these errors, 84 (39%) were possible intrusion errors in which an operand appeared in the error response.

[6] We express our thanks to Mike McCloskey for providing us with the lists of GE's and PS's simple-multiplication errors.

For the same set of problems, GE produced 150 (8.5%) commission errors over 22 blocks of testing. Among these, 35 (23%) involved an operand-error match. The apparently lower rate of operand-error matches for GE may be attributed to the higher rate of cross-operation errors (61 or 39% for GE vs. 14 or 5% for PS), because cross-operation errors generally will be incompatible with the possibility of an intrusion.

For PS, excluding 20 errors made on tie problems (for which the effects of operand position cannot be measured), 64 operand intrusions remained. Position was preserved in 43 cases (67.2%) and not preserved in 21 (32.8%); the observed proportion of .672 is significantly greater than .5 (SE = .0625, z = 2.75, p = .003, one-tailed). For GE, there were 35 operand-error matches on non-tie problems, with position preserved in 25 (71.4%) of the cases, which also is greater than the expected proportion of .50 (SE = .0845, z = 2.54, p = .006). Thus, for both PS and GE, operand order had a significant effect on the frequency of operand-error matches, which confirms that operand intrusions influenced their specific errors. The findings are consistent with the conclusion that one source of error for PS and GE, as well as normals, is interference from numbers directly activated by the problem.

Table 7. Mean Proportion of Errors for PS and GE as a Function of a Positional Match Between a Problem Operand and a Numeral in the Correct Answer (e.g., 6 x 4 = 24 vs. 4 x 6 = 24)

PS			GE		
Match	Mismatch	Diff.	Match	Mismatch	Diff.
.08	.12	.04	.09	.15	.06

Note. Means based on six problems. Data obtained from Sokol et al. (1991), Table 1 (PS) and Table 2 (GE).

We also tested for evidence of operand-answer priming effects comparable to those described previously (see Table 6). Error rates for each problem were reported by Sokol et al. (1991) in their Table 1 for PS (p. 359) and their Table 6 for GE (p. 370). Table 7 presents the mean error rates for PS and GE on the six problems for which an operand matches a number in the same position in the correct answer (e.g., 9 x 5 = 45), and for the six commuted problems that yield a positional mismatch (5 x 9= 45). The rates of errors produced by PS and GE on these problems was approximately equal to that produced by the present group of normal subjects tested under instructions for speed. Again paralleling the results of the normal subjects, both PS and GE showed a tendancy for an accuracy advantage for operand orders in which an operand matches its position in the correct answer. The 6.7% advantage for GE was not significant by a paired-difference t-test [$SE = 5.39, t(5) = 1.24, p > .05$]. The 4.3% advantage for PS did reach significance [$SE = 2.15, t(5) = 2.02, p = .05$ one-tailed].

The evidence of operand intrusions and operand-answer priming in the performance of PS and GE confirm that operand-intrusions and operand-answer priming effects are not phenomena unique to speeded number-fact retrieval by normal adults. Furthermore, the possible influence of operand-reading processes on the performance of PS and GE raises the possibility that their apparent "calculation" deficit (cf. Sokol et al., 1991) may be due, in part, to a failure to control or inhibit number-reading processes. As we discuss in detail later, the possibility of such subtle interactions between components of number comprehension, calculation, and production tasks, make it difficult to localize deficits within a specific number-processing "module."

Discussion

The present results are not easily reconciled with the assumption that number-fact retrieval is mediated only by abstract, format-independent representations contained in a separate calculation module (e.g., McCloskey et al., 1985; Sokol et al., 1989, 1991). The results show that variables theoretically related to retrieval difficulty and interference (Size and Fan) predicted word-digit differences for both RT and errors. These effects emerged over and above encoding effects associated with word length and tie versus non-tie problems. Indeed, the magnitude by which errors increased for word problems (i.e., a 50% increase in commission errors) presents a major challenge to the abstract-code view of number processing. Because numerical and production processes subsequent to comprehension are equivalent for digit and word formats, the locus for the increase in errors must be the comprehension module. That is, comprehension of words must be sufficiently weaker than comprehension of digits to produce a 50% increase in commission errors and a dramatic 138% increase in omission errors. But it seems highly unlikely that comprehension of very familiar number words is in fact so inadequate as to produce such robust effects. Furthermore, the patterns of specific errors, including the numerical distance of errors and the influence of the operands on specific errors varied substantially between format. These results support the encoding-complex view that arithmetic retrieval processes can differ as a function of presentation format.

The results also are consistent with the encoding-complex view that number-reading and number-fact retrieval processes are integrated, which explains operand-answer priming effects (i.e., Table 6), operand-intrusion errors, and naming errors. Moreover, competition among these various sources of activation may sometimes be unresolvable, leading to omission errors. The findings that operand-intrusion errors, pure naming errors, and perhaps operand-answer priming, were stronger effects with word stimuli than with digit stimuli, indicate that these effects are format dependent. One hypothesis is that arithmetic answers are represented as verbal-lexical codes, which are more strongly primed by number words than digits. Under this view, operand-answer priming arises because reading a problem's operands activates verbal number-fact representations that contain the operands. Retrieval is facilitated when operand-answer priming activates the correct problem (as demonstrated in Table 6), or retrieval is disrupted when priming activates a competing problem representation, as evidenced by operand-intrusion errors. These effects are exaggerated with word stimuli because number words are more strongly associated with verbal representations than are digits.

It is not clear how the abstract-modular view can explain such findings, particularly given the central assumption that number-reading mechanisms are assumed to be "functionally independent" of calculation processes (e.g., number-fact representations) and all communication between modules is based on abstract codes. If number-reading processes can directly contribute to specific calculation errors, and also systematically affect correct response times and error rates in arithmetic, then the assumption of functional independence is put in serious doubt.

Under the encoding-complex view effects of format on retrieval occur because arithmetic depends on retrieval of specific codes (e.g., visual and phonological codes) that can be differentially associated with different surface forms. For example, the highly practiced digit format (e.g., 3 x 9 = ?) may support both visuo-spatial and verbal-auditory routes to problem representations. In contrast, number-word stimuli (three x nine = ?) are relatively novel visually, so retrieval of products from number words may be mediated primarily by verbal (e.g., lexical) codes. As mentioned previously, greater dependence on verbal representations of problems potentially explains why word-format retrieval is more susceptible to interference from operand-reading processes. In terms of the network-interference model of number-fact retrieval (Campbell & Oliphant, this volume), the strong visual associations activated by digit-format problems provide an additional basis for discriminating related memory representations. Relative to word problems, therefore, performance on the digit-format problems suffers less from retrieval interference among verbal codes, resulting in faster RTs and fewer errors.

The theory that words and digits can be differentially associated with different internal representations of number-facts also can explain why errors tended to be more numerically distant with word stimuli than digit stimuli. For example, numerical distance or magnitude effects in simple arithmetic may reflect involvement of codes that explicitly represent magnitude (e.g., visuo-spatial analog representations; cf. Dehaene & Cohen (in press); see Campbell & Oliphant, this volume, for a computational model of number-fact retrieval that implements this assumption). Digits may evoke greater involvement of such magnitude codes than number-words because calculations and judgments of magnitude presumably are based more frequently on digit stimuli than number-word stimuli. Thus, number-fact retrieval based on number-words would be less constrained by magnitude information, which would promote errors that are more numerically distant relative to those observed with digits.

Speculatively, reduced involvement of magnitude codes with number-word stimuli potentially explains the observation that the word-format deficit tended to increase with problem size. Campbell and Oliphant (this volume) proposed that problem difficulty generally increases with magnitude because problem representations become less discriminable as absolute magnitude increases. This occurs because the psychophysical scale for magnitude is more compressed for larger values and, consequently, larger-number problems encounter more retrieval interference from neighbouring problems. This relatively poor discrimination of magnitude for larger problems would be exaggerated with number-words, relative to digits, if number words activate generally weaker or diffuse magnitude information.

In summary, multiplication performance varies with surface form in ways that are not easily reconciled with the assumption that number-fact retrieval is based only on abstract, format-independent processes. We have demonstrated a variety of phenomena that strongly suggest that format directly affects retrieval processes, rather than only encoding or production processes. In addition to overall effects on speed and accuracy of performance, we demonstrated that format also affected the relative frequencies of various types of errors, affected changes in the numerical distance of errors, and altered the influence of operand-reading processes on number-fact retrieval. These findings support the encoding-complex view that number-fact retrieval is based on multiple types of specific codes (e.g., verbal-lexical or visuo-spatial codes), and that these specific codes can differentially affect retrieval performance as a function of presentation format.

Is the hypothesis of abstract number codes necessary?

The foregoing evidence for format-specific phenomena, and hence for specific codes, does not disprove the hypothesis of abstract codes, and that is not our intent. We simply have documented evidence that specific codes appear to be necessary to explain many aspects of number processing. Nonetheless, given the evidence that both elementary aspects of number meaning and more complex numerical tasks involve multiple, specific codes, the question arises as to whether a more abstract level of representation is also necessary.

One source of evidence against the assumption that number-processing requires a special kind of abstract, semantic code, as proposed by McCloskey et al., comes from research on memory retrieval for non-numerical stimuli. Graham and Campbell (in press) had children in grades 3 and 4 memorize *alphaplication* facts; arithmetic-like memory items composed of letters instead of numbers (e.g., A,I = x; I,U = b). The children were quite successful at memorizing 25 such items in two training sessions (70% accuracy under instructions for speed), and memory for alphaplication problems paralleled memory for arithmetic facts in many respects. Among other similarities, alphaplication results showed a) a large performance advantage for tie (e.g., E,E = j) over non-tie problems (E,I = p), b) most errors involved answers from the correct "alpha-table," c) response times and error rates were strongly correlated across problems, d) performance on commuted pairs was highly correlated, and e) the correct answers to poorly learned problems tended to be the most common error responses. These parallels suggest that retrieval of alphaplication and multiplication facts is similar with respect to at least some of the basic memory processes involved.

Unlike multiplication, however, the operands of alphaplication problems were not systematically or meaningfully related to answers; thus, learning alphaplication simply involved memorizing arbitrary associations among combinations of letters. The alphaplication data therefore provide a sufficiency proof that memory for arithmetic-like stimuli does not require an underlying abstract semantic code such as that proposed by McCloskey et al. Magnitude representations and knowledge of other numerical relations are, of course, semantically relevant in genuine arithmetic (cf. Campbell & Oliphant, this volume), but the alphaplication data demonstrate that arbitrary associations provide sufficient basis for performance, and there seems no good reason to reject the assumption that such arbitrary associations among specific symbols (e.g., digits and words) also contribute to retrieval of arithmetic facts and other numerical tasks.

One putative type of evidence supporting the assumption of abstract codes is similarity when numbers are processed in different stimulus formats and modalities (e.g., Sokol et al., 1989; Sokol et al., 1991). Indeed, in the preceding study, performance on digit and word problems was highly correlated. Such findings, however, do not constitute direct evidence for abstract codes. One possible explanation for these similarities is that a common code is used in number processing, but that the code is specific rather than abstract. That is, subjects may translate all stimulus codes into number word or visual digit format, so that calculations are produced predominantly via the same specific code. A related explanation is that the various specific numerical codes become so strongly associated as to activate one another with a high probability every time any one of the codes is activated, unless special conditions are implemented to interfere selectively with specific modalities. Still another possibility is that many features of the associative structure emerge from learning and contextual experiences that would develop and operate similarly in different stimulus modes. That is, interfering associations, effects of priming, and related mechanisms may operate similarly whether number words or visual digits constitute the nodes in the associative network. Thus, evidence for unique features of number processing with different surface forms or modalities provides positive evidence for specific codes, but the converse is not necessarily true.

Modular versus integrated views of cognitive number processing

In the abstract-modular theory, the presumed ability to localize the source of a number-processing impairment within one of the hypothetical comprehension, calculation, or production systems provides the primary evidence for the reality of these systems. We have pointed out, however, that this ability depends on the assumption that a single type of abstract number code mediates all computation and production tasks. Only this assumption permits performance on one task to be used as a control condition for the integrity of processes assumed to be shared with another task (e.g., sum verification may be assumed to involve the comprehension and calculation processes required for sum production). The abstract codes, in effect, define the boundaries that separate the three hypothetical systems: The comprehension system converts different surface forms of numbers into abstract semantic codes that then are passed on to the calculation or production systems, and the production system converts abstract codes received from the calculation or comprehension modules back into specific number formats (e.g., written digits or number words).

Once it is allowed that different facets of number processing can involve multiple, modality-specific codes, the boundaries among number comprehension, calculation, and production become blurred. For example, as pointed out earlier, the apparent influence of operand intrusions on the multiplication errors produced by PS and GE (Sokol et al., 1991) suggests that their apparent calculation deficit could arise, in part, from abnormal interference by number-reading processes (a "production" deficit?). Conversely, Campbell and Clark (1988) demonstrated that the number-reading errors (e.g., stating "six" for "2") produced by McCloskey et al.'s (1986) patient, HY, were predicted by visual and arithmetic similarity (numerical nearness and odd-even agreement). Although such results can be reconciled with a pure production deficit, they are also consistent with the hypothesis that HY's deficit was due to a failure to inhibit visually similar codes (a "comprehension" deficit?) or a failure to inhibit arithmetically related responses (a "calculation" deficit?). That such factors collectively affect performance challenges not only the logic by which a deficit is localized within a particular subsystem, but, more generally, the view that number processing can be carved neatly into a small number of independent systems.

That the boundaries among number comprehension, calculation, and production are not drawn as sharply as McCloskey et al. (1985; Sokol et al., 1989) propose is illustrated by a consideration of counting skills, and how counting is related to calculation and number-production. Although nothing specific has been claimed about "where" counting occurs in the abstract-modular number-processing system, Sokol et al. (1989) state that they do not consider counting to be a normal function of the calculation system (p. 108). Furthermore, the verbal production subsystem of the McCloskey et al. model is implicated by much research that indicates that counting relies on phonological and articulatory codes (e.g., Healy & Nairne, 1985; Logie & baddeley, 1987; Seron & Deloche, 1987). But localizing counting processes somewhere "outside of" the calculation system raises a problem for the modularity assumption because some of the tasks assumed by McCloskey et al. (1985) and McCloskey et al. (1986) to involve the calculation system (e.g., production and verification of arithmetic facts) can in fact be performed using counting strategies instead of retrieval. There is evidence that even normal, educated adults sometimes use such procedural strategies as counting for simple addition (Geary & Wiley, 1991). Thus, "calculation" tasks may be performed using strategies that do not involve the hypothetical calculation system. Similarly, there is evidence that arithmetic-verification (e.g., 4 + 8 = 15, true or false?), which is used to assess the integrity of components of the calculation system (McCloskey et al., 1985; McCloskey et al., 1986), is sometimes performed by judging the magnitude (Stazyk, Ashcraft & Hamann, 1982) or odd-even status of the presented answer (e.g., Krueger, 1986; Krueger & Hallford, 1984). As mentioned previously, however, Sokol et al. (1989) suggested that the processing of magnitude and odd-even status could be external to the calculation module. Thus, again, "calculation" performance may depend on resources external to the putative calculation system.

Such considerations are one source of motivation for the encoding-complex view that comprehension, calculation, and production are integrated processes rather than separate modules. Performance on the verification task illustrates further the integrated nature of comprehension, calculation, and verbal production. Sum verification, a presumed "calculation" task, can be solved by assessing the relative magnitude of the problem and the presented answer. Such magnitude judgments (e.g., identify the larger number), however, could be based on counting string associations ("six" must be greater than "five" because it occurs later in counting), suggesting that the verbal production processes involved in counting potentially mediate verification. On some verification trials, subjects may generate an answer for the problem and check it against the presented answer (cf. Campbell, 1987; Widaman, Geary, Cormier & Little, 1989). This comparison process could involve verbal representations of the corresponding number words (although comparisons of visual representations are also possible), and an error in the comparison or number-word production process could lead to an incorrect verification response. Furthermore, Campbell (1987; 1991) showed that brief pre-exposure (200-300 ms) of the correct answer sharply reduces retrieval errors on simple arithmetic problems, and this facilitative priming effect may well operate when the correct answer is presented in a verification task. Given such considerations, successful verification may provide only limited information about the integrity of the arithmetic retrieval processes involved in generating an answer.

The various examples that we have considered illustrate, not only that it is ambiguous which "modules" a given task implicates, but also that interpretation of dissociations between tasks will depend critically on understanding how different strategies and knowledge structures influence performance on those tasks. Naturally, many tasks will share some processes in common, but the potential for processing variability within and across tasks strongly recommends against uncritically applying the methodological strategy of using performance on one task as a control for components of another task (cf. Dunn & Kirsner, 1988).

The relation between arithmetic and number naming

Further difficulties associated with separating functions of number comprehension, calculation, and production arise from considering the role of arithmetic in number naming. In the abstract-modular framework proposed by McCloskey et al. (1985), arithmetic processes (e.g., knowledge of simple arithmetic operations and facts) are functionally separated from language processes (i.e., syntactic and lexical mechanisms) contained in the production system. This separation of arithmetic and linguistic processes seems to ignore the intimate relationship between number-naming systems and arithmetic relations. Although the names for "primitive numbers" (i.e., numbers which are directly named in a given language) are often arbitrary, in some languages the basic number names are related to objects or events that imply the corresponding quantity (e.g., the word for "five" may be based on the word for "hand"; Fuson & Kwon, 1990). Furthermore, in many, if not all, languages (cf. Boden, 1988), the production rules by which a given numeral (e.g., 2226) is converted to a number-word sequence (two-thousand two-hundred and twenty-six) correspond to arithmetic relationships (i.e., 2 x 1000 + 2 x 100 + 20 + 6). The intrinsic knowledge of arithmetic entailed by number naming systems is also demonstrated by the various ways in which the same number can be named. For example, the number 2200 may be stated as "two thousand two hundred" or "twenty-two hundred," depending on which arithmetic relationships are used to parse and interpret the string of digits. These observations suggest that different linguistic number-naming conventions can involve different semantic specifications or, put another way, that linguistic form potentially determines the underlying semantic representation for numbers and how calculations are performed.

Indeed, there is empirical evidence that number representation and calculation skills are at least partially determined by linguistic structure. For example, there is a close connection between the arithmetic structure of number-naming systems and how children represent number. Fuson and Kwon (1990) review studies showing that children represent quantities using tokens that correspond to linguistic structure. Asian number-naming systems usually quantify the tens positions explicitly (e.g., 34 is named "three ten four" in Chinese), whereas European languages do not explicitly express the decade position as a multiple of ten. Children who learn a system that names the tens are more likely to represent 34, for example, with three tens tokens and four units tokens, whereas age-matched children learning a system that does not name the tens are more likely to count out 34 units. Furthermore, these differences in number-naming systems also affect how children learn to perform single-digit and multi-digit addition and subtraction (Fuson and Kwon, 1990). Such observations challenge a model that sharply separates functions of calculation from the grammatical processes of verbal number production.

Autonomous procedures versus modularity

The encoding-complex view rejects a strong form of modularity, but it does not follow that there are no differentiated aspects of number processing. For example, the "syntactic frame" model for verbal number-production proposed by McCloskey et al. (1986) provides a plausible account of how numbers presented as digits are mapped on to verbal number names. Note, however, that this component of verbal number production requires neither the assumption of abstract codes nor the assumption that number production mechanisms occupy a separate system. Despite this disclaimer, it is certainly possible that the semantic relations entailed in converting a string of digits to the appropriate number name become *proceduralized* with practice (Anderson, 1982), which implies that the underlying semantic structure would no longer need to be explicitly processed in the context of a simple naming task. This specific skill, once acquired, may therefore involve a relatively autonomous cognitive routine. Nonetheless, the encoding-complex view stills permits number production routines to vary across tasks because different tasks emphasize different representational media. For example, when the digit string "1026" is presented for naming, the computation of place value (i.e., the function of the syntactic frame) may be based directly on an analysis of the visual form of the stimulus. If, instead, one generates the number name for the total quantity represented by one "thousands" token, two "tens" tokens, and six "ones" tokens, a visual frame is not directly available from the input structure, and a different process for generating the correct grammatical structure of the output may be required.

In summary, the assumption that number comprehension, calculation, and production involve independent cognitive subsystems, as proposed by McCloskey et al., appears to oversimplify the relationships among components of number processing. According to the encoding-complex view, number comprehension, calculation and production are terms that refer to somewhat distinct collections of tasks that vary in information or response requirements, but which, nonetheless, often draw on common verbal and visual networks of associative relations. The encoding-complex view that number skills are based on multiple forms of internal representations and can be realized in many ways, advises directly against an approach that tends to ignore this plasticity in favor of a relatively simple taxonomy of number processing subsystems. The better part of understanding performance in number processing tasks may be in understanding its diversity and complexity.

Conclusions

The encoding-complex view maintains that number-processing tasks can be understood in terms of retrieval and comparison of elementary visual and verbal codes. That is, these elementary associative codes and processes are the building blocks out of which number processing skills and routines are constructed. Relatively simple tasks such as counting, comparison of magnitude and other elementary number features, generation and verification of basic number facts, as well as some components of written and spoken number production are already amenable to a detailed analysis in terms of elementary associations. Although the modality-specific codes that provide the representational foundation for number processing do not in and of themselves explain the various numerical operations that people can perform (e.g., comparisons of magnitude, multiplication and other basic operations, complex calculations, sophisticated aspects of mathematical reasoning, and so on), these and other competencies must ultimately be explained in terms of basic associative mechanisms. Our ultimate objective is to identify the principles by which excitatory and inhibitory connections among various specific representations cause number processing phenomena to emerge from the collective activity of associative networks (e.g., Campbell & Oliphant, this volume).

In our approach we also seek to identify common mechanisms that underlie numerical and nonnumerical domains. We doubt that the human brain involves number-processing mechanisms that are completely independent of non-numerical cognitive functions, although some unique features may emerge from the particular ways in which mechanisms are combined in the area of number. Thus, we believe that number skills do not comprise a separable cognitive "system," and indeed there is a substantial research literature demonstrating continuity of numerical and non-numerical skills (see Clark & Campbell, 1991). It seems unlikely, for example, that the brain has a "power of ten" mechanism that is uniquely numerical in nature (cf. the abstract codes assumed by McCloskey et al., 1986). Instead we suspect that the "power of ten" emerges from some as yet determined collection of basic quantity representations and associations, perhaps involving the use of place or position at some level. We have also argued that the quantity representations are themselves modality specific, which makes them not qualitatively distinct from non-number representations. Note that our position still allows some emergent aspects of number processing to be unique. Whatever the underlying mechanism is for "powers of ten," for example, it is unlikely that the primitives behave collectively in an identical way in non-number domains.

The encoding-complex theory in its current form does not constitute a complete model of number processing, and we acknowledge that, at present, there is uncertainty regarding the taxonomy of representational codes that will be necessary and sufficient for a complete encoding-complex theory of cognitive number skills. Furthermore, without detailed models of specific tasks, precise predictions often are not possible because of the complex associative processes by which the various interconnected codes potentially activate and inhibit one another. Despite such uncertainty, the encoding-complex approach does represent a substantive and clear alternative to the global assumptions of abstract-codes and modularity of number functions. Although proponents of the abstract-modular view have dismissed the encoding-complex approach as "vacuous" (Sokol et al., 1989), our theory that semantic number representations may be modality- and format-specific is no more inherently vacuous than the assumption that number representations are wholly abstract and divorced from modality and surface characteristics (see Clark & Campbell, 1991, for a critique of the abstract codes proposed by McCloskey et al., 1986). Indeed, the issue of abstract versus format-specific semantic codes is far from resolved in research on picture and word processing and in research on semantic representation in multilinguals (e.g., Clark, 1987; Vaid, 1988; Glucksberg, 1984). Furthermore, as the present experimental findings have demonstrated, there is considerable evidence that calculation processes vary with surface form, and such findings represent a strong challenge to the abstract-code hypothesis.

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